Solution 8

Supplementary Problems

1. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be on $[a, b]$ and $[\alpha, \beta]$ respectively that describe the same curve C. It has been shown that there exists some φ maps $[a, b]$ one-to-one onto $[\alpha, \beta]$, $\varphi'(t) > 0$, such that $\mathbf{r}_2(\varphi(t)) = \mathbf{r}_1(t)$ when both parametrization runs in the same direction. When they runs in different direction, $\varphi'(t) < 0$. Using this fact to prove that in both cases,

$$\int_a^b f(\mathbf{r}_1(t))|\mathbf{r}_1'(t)|\, dt = \int_\alpha^\beta f(\mathbf{r}_2(z))|\mathbf{r}_2'(z)|\, dz \ .$$

In other words, the line integral

$$\int_C f\, ds$$

is independent of the choice of parametrization with the same or opposite direction.

Solution. Differentiating the relation $\mathbf{r}_2(\varphi(t)) = \mathbf{r}_1(t)$ and using the chain rule, we get

$$\mathbf{r}_2'(z)\varphi'(t) = \mathbf{r}_1'(t) \ ,$$

so

$$|\mathbf{r}_2'(z)||\varphi'(t)| = |\mathbf{r}_1'(t)| \ , \quad z = \varphi(t).$$

We have

$$\begin{aligned} \int_{\mathbf{r}_2} f\, ds &= \int_\alpha^\beta f(\mathbf{r}_2(z))|\mathbf{r}_2'(z)|\, dz \\ &= \int_\alpha^\beta f(\mathbf{r}_2(z))\frac{|\mathbf{r}_1'(t)|}{|\varphi'(t)|}\, dz \ . \end{aligned}$$

When $\varphi(a) = \alpha, \varphi(b) = \beta$ and $\varphi' > 0$, by the change of variables formula, we continue

$$\begin{aligned} &= \int_a^b f(\mathbf{r}_2(\varphi(t)))\frac{|\mathbf{r}_1'(t)|}{|\varphi'(t)|}\varphi'(t)\, dt \\ &= \int_a^b f(\mathbf{r}_1(t))|\mathbf{r}_1'(t)|\, dt \\ &= \int_{\mathbf{r}_1} f\, ds \ . \end{aligned}$$

On the other hand, when $\varphi(a) = \beta, \varphi(b) = \alpha$ and $\varphi' < 0$, we have

$$\begin{aligned} &= \int_b^a f(\mathbf{r}_2(\varphi(t)))\frac{|\mathbf{r}_1'(t)|}{-\varphi'(t)}\varphi'(t)\, dt \\ &= \int_a^b f(\mathbf{r}_1(t))|\mathbf{r}_1'(t)|\, dt \\ &= \int_{\mathbf{r}_1} f\, ds \ . \end{aligned}$$

The same result holds.